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Polarized angular distributions in the decays of the ψ' charmonium state directly produced in unpolarized proton–antiproton collisions

A.W.K. Mok[1,a], K.J. Sebastian[2,b]

[1] Department of Physics, Hong Kong Baptist University, Kowloon Tong, Hong Kong, P.R. China
[2] Department of Physics, University of Massachusetts at Lowell, Lowell, Massachusetts 01854, USA

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Abstract. Using the helicity formulism, we calculate the combined angular distribution function of the polarized gamma photons and electron in the cascade process $\bar{p}p \to \psi' \to \chi_J + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to e^+ + e^- + \gamma_2 + \gamma_1$ $(J = 0, 1, 2)$, when $\bar{p}$ and p are unpolarized. We also present the partially integrated angular distribution functions in different cases. Our results show that by measuring the two-particle angular distribution of γ_1 and γ_2 and that of γ_2 and e^- with the polarization of either one of the two particles, one can determine the relative magnitudes as well as the relative phases of the helicity amplitudes in the radiative decay processes $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$.

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1 Introduction

Model-independent studies of the angular distribution in the decays of charmonium states directly produced in $\bar{p}p$ collisions can give us valuable information on the true dynamics of the charmonium system. The helicity formulism, first introduced by Jacob and Wick in 1959 [1], allows us to separate the consequences of the quantum mechanics and symmetry from those of the detailed dynamics in the angular distribution. The angular-momentum helicity amplitudes can be defined [1, 2] in such a way that all the dynamics is contained in them. The expression for the angular distribution is given in terms of these helicity amplitudes and the Wigner D^J functions. So, in principle, the experimental determination of the angular distribution of the final decay products of the charmonium allows us to obtain important information on the relative magnitudes as well as the relative phases of these helicity amplitudes. In order to obtain the overall absolute magnitudes one also has to measure the branching ratios of the decays as well as the total decay-width or the lifetime of the charmonium states. All of these measurements for the ψ' charmonium state will be prospectively obtained from the E835 experiment at the Fermilab [3] and the planned PANDA experiment at FAIR [4], which study charmonium spectroscopy in $\bar{p}p$ annihilation.

In a previous paper [5], it is shown that by measuring the combined angular distribution of the two photons and of the electron, regardless of their polarizations, in the sequential process originating from unpolarized $\bar{p}p$ collisions, namely, $\bar{p}p \to \psi' \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_1 + \gamma_2$ $(J = 0, 1, 2)$, one can extract the relative magnitudes as well as the cosines of the relative phases of all the angular-momentum helicity amplitudes in the radiative decay processes $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. The sines of the relative phases are not determined uniquely. By including the measurement of the polarization of one of the decay particles, one may be able to also obtain unambiguously the sines of these relative phases and thus complete information on all the angular-momentum helicity amplitudes in the radiative decay processes. So in this paper we calculate the angular distributions of the final stable decay products, γ_1, γ_2 and e^-, with the determination of the polarization of any one of the three particles in the above cascade process when $\bar{p}$ and p are unpolarized. Our final expressions for the angular distribution functions are valid in the $\bar{p}p$ center-of-mass frame and they are written as sums of terms involving products of the Wigner D^J functions whose arguments are the angles representing the directions of the final electron and of the two photons. The coefficients in these expansions are functions of the angular-momentum helicity amplitudes in the different individual processes of the above cascade process. Once the combined angular distribution of γ_1, γ_2 and e^- and the polarization of any one of the particles in unpolarized $\bar{p}p$ collisions are experimentally measured, our expressions

[a] e-mail: wkmok@hkbu.edu.hk
[b] e-mail: kunnat_sebastian@uml.edu

will enable one to calculate the relative magnitudes as well as the relative phases of all the angular-momentum helicity amplitudes in the two radiative decay processes mentioned above for all values of J. Therefore one can get complete information on these helicity amplitudes. In addition, one can also determine the relative magnitudes of the angular-momentum helicity amplitudes in the processes $\bar{p}p \to \psi'$ and $\psi \to e^+e^-$. Our results on the partially integrated angular distributions where the combined angular distribution function of γ_1, γ_2 and e^- is integrated over the directions of one or two particles are quite interesting. They show that by measuring the two-particle angular distribution of γ_1 and γ_2 as well as that of γ_2 and e^- with the polarization of either one of the two particles, one can get as much information on the helicity amplitudes as one obtained from measuring the combined angular distribution of γ_1, γ_2 and e^- with the polarization of any one particle.

The format of the rest of the paper is as follows. In Sect. 2, we give the calculation for the combined angular distribution with polarization determination of the electron and of the two photons in the cascade process $\bar{p}p \to \psi' \to \chi_J + \gamma_1 \to (\psi + \gamma_2) + \gamma_1 \to (e^+e^-) + \gamma_1 + \gamma_2$ $(J = 0, 1, 2)$, when $\bar{p}$ and p are unpolarized. We then show how the measurement of this combined angular distribution of polarized γ_1, γ_2 and e^- enables us to obtain complete information on the helicity amplitudes in the two radiative decay processes $\psi' \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. We also present three different results for the combined angular distribution, in which the polarization of only one of the three particles, γ_1, γ_2 and e^-, is measured. In Sect. 3, we present the results for the partially integrated angular distributions in different cases where the combined angular distribution function of the three particles is integrated over the directions of one or two particles. These results can all be expressed in terms of the orthogonal spherical harmonic functions. We point out how the measurement of these partially integrated angular distributions will again give us complete information on all the helicity amplitudes in the two radiative decay processes. Finally, in Sect. 4, we make some concluding remarks.

2 Calculations for the combined angular distribution of polarized γ_1, γ_2 and e^-

We consider the cascade process, $p(\bar{\lambda}_1) + p(\lambda_2) \to \psi'(\delta) \to \chi_J(\nu) + \gamma_1(\mu) \to \psi(\sigma) + \gamma_2(\kappa) + \gamma_1(\mu) \to e^-(\alpha_1) + e^+(\alpha_2) + \gamma_2(\kappa) + \gamma_1(\mu)$ $(J = 0, 1, 2)$, in the $\bar{p}p$ center-of-mass frame or the ψ' rest frame, where J is the angular momentum of the χ resonance and the Greek symbols after the particle symbols represent their helicities except for the stationary ψ' resonance, in which case the symbol δ represents the z component of the angular momentum. We choose the z axis to be in the direction of motion of χ_J in the ψ' rest frame. The x and y axes are arbitrary in our discussions. The experimentalists can choose them according to his or her convenience. The probability amplitude for the above cascade process can be written as a product of the matrix elements for the individual sequential processes. Since only the helicities of the initial and the final particles, namely, λ_1, λ_2, μ, κ, α_1 and α_2, are observed, we write the probability amplitude for the cascade process in the ψ' rest frame as

$$T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \sum_{\delta,\sigma}^{-1,0,+1} \sum_{\nu}^{-J\to+J} {}_{\psi'}\langle\psi'(\delta)|B|\bar{p}(\lambda_1), p(\lambda_2)\rangle_{\psi'} \times {}_{\psi'}\langle\chi_J(\nu), \gamma_1(\mu)|A|\psi'(\delta)\rangle_{\psi'} \times {}_{\psi'}\langle\psi(\sigma), \gamma_2(\kappa)|E|\chi_J(\nu)\rangle_{\psi'} \times {}_{\psi'}\langle e^-(\alpha_1), e^+(\alpha_2)|C|\psi(\sigma)\rangle_{\psi'} . \quad (1)$$

We sum over the helicities and the spin indices of the unobserved intermediate particles in (1). The symbols B, A, E and C represent the appropriate transition operators. The subscript ψ' attached to the bra or the ket vector indicates that each individual matrix element is evaluated in the ψ' rest frame. In the first two matrix elements the ψ' rest frame is the same as the c.m. frame of the two particles. In the last two matrix elements $\langle\psi\gamma_2|E|\chi_J\rangle$ and $\langle e^-e^+|C|\psi\rangle$ this is not the case. To avoid confusion, we should clarify what we mean by the two-particle helicity states when they are not in their c.m. frame. For example, the two-particle state $|\psi(\sigma), \gamma_2(\kappa)\rangle_{\psi'}$ defined in the ψ' rest frame, which is not the c.m. frame of ψ and γ_2, has the following meaning. First construct the two-particle helicity state $|\psi(\sigma), \gamma_2(\kappa)\rangle_{\chi_J}$ in the χ_J rest frame (which is the same as the c.m. frame of ψ and γ_2) according to the usual conventions [2] with ψ and γ_2 having equal and opposite momenta and helicities σ and κ, respectively. Then

$$|\psi(\sigma), \gamma_2(\kappa)\rangle_{\psi'} = U_\Lambda(\psi', \chi_J)|\psi(\sigma), \gamma_2(\kappa)\rangle_{\chi_J} , \quad (2)$$

where $U_\Lambda(A, B)$ is the unitary operator corresponding to the Lorentz transformation $\Lambda(A, B)$ which takes the system from the Lorentz frame where B is at rest to the Lorentz frame where A is at rest. It is important to clarify this point since in general ψ and γ_2 do not have definite helicities in the ψ' rest frame. A similar meaning also holds for the two-particle state $|e^-(\alpha_1), e^+(\alpha_2)\rangle_{\psi'}$.

Let us now consider the matrix elements in (1) one by one. First,

$${}_{\psi'}\langle\psi'(\delta)|B|\bar{p}(\lambda_1), p(\lambda_2)\rangle_{\psi'} = \langle 1\delta|B|p(\theta, \phi); \lambda_1\lambda_2\rangle , \quad (3)$$

where $\langle 1\delta|$ is the one-particle helicity state, or the angular-momentum state, of ψ' in its own rest frame and $p(\theta, \phi)$ is the magnitude of the c.m. momentum of $\bar{p}$, which is taken to be in the direction (θ, ϕ) in the coordinate system we have chosen. Using the usual expansion [2] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we find [6]

$${}_{\psi'}\langle\psi'(\delta)|B|\bar{p}(\lambda_1), p(\lambda_2)\rangle_{\psi'} = \sqrt{\frac{3}{4\pi}} B_{\lambda_1\lambda_2} D^1_{\delta\lambda}(\phi, \theta, -\phi) , \quad (4)$$

where

$$\lambda = \lambda_1 - \lambda_2 . \quad (5)$$

and $B_{\lambda_1\lambda_2}$ are the angular-momentum helicity amplitudes.

Similarly, the matrix element for the process $\psi' \to \chi_J + \gamma_1$ with χ_J and γ_1 moving along the $+z$ and $-z$ directions, respectively, can be written as

$$\begin{aligned} {}_{\psi'}\langle \chi_J(\nu), \gamma_1(\mu)|A|\psi'(\delta)\rangle_{\psi'} &= \langle p_{\chi_J}(0,0); \nu\mu|A|1\delta\rangle \\ &= \sqrt{\frac{3}{4\pi}} A^J_{\nu\mu} D^{1*}_{\delta,\nu-\mu}(0,0,0) \\ &= \sqrt{\frac{3}{4\pi}} A^J_{\nu\mu} \delta_{\delta,\nu-\mu} , \end{aligned} \tag{6}$$

where $p_{\chi_J}(0,0)$ is the magnitude of the momentum of χ_J along the z axis in the ψ' rest frame and the $A^J_{\nu\mu}$ are the angular-momentum helicity amplitudes for this process.

Next we notice that the matrix elements for the process $\chi_J \to \psi + \gamma_2$ in the ψ' and the χ_J rest frames are equal. That is,

$$\begin{aligned} &{}_{\psi'}\langle \psi(\sigma), \gamma_2(\kappa)|E|\chi_J(\nu)\rangle_{\psi'} \\ &\quad = {}_{\chi_J}\langle \psi(\sigma), \gamma_2(\kappa)|U^\dagger_\Lambda(\psi', \chi_J) E U_\Lambda(\psi', \chi_J)|\chi_J(\nu)\rangle_{\chi_J} \\ &\quad = {}_{\chi_J}\langle \psi(\sigma), \gamma_2(\kappa)|E|\chi_J(\nu)\rangle_{\chi_J} . \end{aligned} \tag{7}$$

In (7) we have used the fact that the transition operator E is invariant under Lorentz transformations:

$$U^\dagger_\Lambda E U_\Lambda = E . \tag{8}$$

Using (7) we can now write

$${}_{\psi'}\langle \psi(\sigma), \gamma_2(\kappa)|E|\chi_J(\nu)\rangle_{\psi'} = {}_{\chi_J}\langle p'(\theta', \phi'); \sigma\kappa|E|J\nu\rangle_{\chi_J} , \tag{9}$$

where $p'(\theta', \phi')$ is the magnitude of the ψ three-momentum in the χ_J rest frame or the ψ–γ_2 c.m. frame. Moreover, in this frame, the index ν is the z-component of the total angular momentum of χ_J. Again using the expansion of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we obtain

$$\begin{aligned} &{}_{\psi'}\langle \psi(\sigma), \gamma_2(\kappa)|E|\chi_J(\nu)\rangle_{\psi'} = \\ &\sqrt{\frac{2J+1}{4\pi}} E^J_{\sigma\kappa} D^{J*}_{\nu,\sigma-\kappa}(\phi', \theta', -\phi') , \end{aligned} \tag{10}$$

where $E^J_{\sigma\kappa}$ are the angular-momentum helicity amplitudes for the process.

For the matrix element of the final process $\psi(\sigma) \to e^-(\alpha_1) + e^+(\alpha_2)$ the situation is more involved. We have

$$\begin{aligned} &{}_{\psi'}\langle e^-(\alpha_1), e^+(\alpha_2)|C|\psi(\sigma)\rangle_{\psi'} \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|U^\dagger_\Lambda(\psi', \psi) C U_\Lambda(\psi', \chi_J) \\ &\qquad \times U_\Lambda(\chi_J, \psi)|\psi(\sigma)\rangle_\psi \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|U^\dagger_\Lambda(\psi', \psi) C U_\Lambda(\psi', \psi) U^\dagger_\Lambda(\psi', \psi) \\ &\qquad \times U_\Lambda(\psi', \chi_J) U_\Lambda(\chi_J, \psi)|\psi(\sigma)\rangle_\psi \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|C U^\dagger_\Lambda(\psi', \psi) \\ &\qquad \times U_\Lambda(\psi', \chi_J) U_\Lambda(\chi_J, \psi)|\psi(\sigma)\rangle_\psi . \end{aligned} \tag{11}$$

In the first equality of (11) we have made use of the fact that the single-particle state $|\psi(\sigma)\rangle_{\psi'}$ was also part of the two-particle helicity state in (10). It was obtained by successively performing two unitary operations corresponding to two Lorentz transformations, the first taking the ψ state from its rest frame to the χ_J rest frame and the second taking it from the χ_J rest frame to the ψ' rest frame. In the last equality of (11) we now make use of the fact that

$$U_\Lambda(\psi', \chi_J) U_\Lambda(\chi_J, \psi) = U_\Lambda(\psi', \psi) U_{R_\mathrm{W}} , \tag{12}$$

where U_{R_W} is a unitary operator corresponding to a pure rotation, usually called "Wigner rotation". Using (12) and the unitarity of U_Λ, (11) now leads to

$$\begin{aligned} &{}_{\psi'}\langle e^-(\alpha_1), e^+(\alpha_2)|C|\psi(\sigma)\rangle_{\psi'} \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|C U_{R_\mathrm{W}}|\psi(\sigma)\rangle_\psi \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|U_{R_\mathrm{W}} U^\dagger_{R_\mathrm{W}} C U_{R_\mathrm{W}}|\psi(\sigma)\rangle_\psi \\ &\quad = {}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|U_{R_\mathrm{W}} C|\psi(\sigma)\rangle_\psi , \end{aligned} \tag{13}$$

since

$$U^\dagger_{R_\mathrm{W}} C U_{R_\mathrm{W}} = C . \tag{14}$$

Using the expansion of the two-particle helicity state in terms of the angular-momentum states, we can write the right-hand side of (13) as

$$\begin{aligned} &{}_{\psi}\langle e^-(\alpha_1), e^+(\alpha_2)|U_{R_\mathrm{W}} C|\psi(\sigma)\rangle_\psi \\ &\quad = \sqrt{\frac{3}{4\pi}} D^{1*}_{\delta\alpha}(R^{-1}_\mathrm{W} \hat{e}_\psi) C_{\alpha_1\alpha_2} \\ &\quad = \sqrt{\frac{3}{4\pi}} C_{\alpha_1\alpha_2} D^{1*}_{\delta\alpha}(\phi'', \theta'', -\phi'') , \end{aligned} \tag{15}$$

where

$$\alpha = \alpha_1 - \alpha_2 , \tag{16}$$

$\hat{e}_\psi$ is a unit vector in the direction of e^- three-momentum in the ψ rest frame, R_W is the (3×3) rotation matrix and $C_{\alpha_1\alpha_2}$ are the angular-momentum helicity amplitudes for the process. We should mention here that if the electron–positron pair is created by a virtual photon via $\bar{q}q \to \gamma \to e^+e^-$, the helicity zero amplitude C_{++} or C_{--} is of the order m/E when compared to the helicity 1 amplitude C_{+-} or C_{-+}. Since $E \cong M_\psi/2$ where M_ψ is the rest mass of ψ, $m/E \cong 3.3 \times 10^{-4}$ and the helicity zero amplitude is relatively negligible.

The Wigner-rotated unit vector $R^{-1}_\mathrm{W} \hat{e}_\psi$ can be obtained in the following way. If R represents the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above, then, from the definition of U_{R_W} in (12),

$$R = \Lambda^{-1}(\psi', \psi) \Lambda(\psi', \chi_J) \Lambda(\chi_J, \psi) , \tag{17}$$

where the Λ are the (4×4) Lorentz transformation matrices. Now we note that the electron is highly relativistic in the ψ rest frame and its four-momentum vector p_{e_ψ} can be represented to a very good approximation by

$$p_{e_\psi} = \frac{M_\psi}{2}(1, \hat{e}_\psi) , \tag{18}$$

and therefore

$$\begin{aligned} R^{-1}p_{e_\psi} &= \Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}(\psi',\chi_J)\Lambda(\psi',\psi)p_{e_\psi} \\ &= \Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}(\psi',\chi_J)\Lambda(\psi',\psi)\Lambda^{-1}(\psi',\psi)p_{e_{\psi'}} \\ &= \Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}(\psi',\chi_J)p_{e_{\psi'}} \,. \end{aligned} \tag{19}$$

In (19) the four-momentum of e^- in the ψ' rest frame is given by

$$p_{e_{\psi'}} = E_{e_{\psi'}}(1,\hat{e}_{\psi'})\,, \tag{20}$$

where $E_{e_{\psi'}}$ is the relativistic energy of e^-, and $\hat{e}_{\psi'}$ is the unit vector in the direction of the three-momentum of e^- in the ψ' rest frame. From (18) we also have

$$R^{-1}p_{e_\psi} = \frac{M_\psi}{2}\left(1, R_{\mathrm{W}}^{-1}\hat{e}_\psi\right). \tag{21}$$

Combining (19)–(21) we get

$$\frac{M_\psi}{2}\left(1, R_{\mathrm{W}}^{-1}\hat{e}_\psi\right) = \Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}(\psi',\chi_J)E_{e_{\psi'}}(1,\hat{e}_{\psi'})\,. \tag{22}$$

The spatial part of the right-hand side of (22) gives, within a normalization factor, the Wigner-rotated unit vector $\hat{\tilde{e}} = (R_{\mathrm{W}}^{-1}\hat{e}_\psi)$ in terms of the angles $(\tilde{\theta}'', \tilde{\phi}'')$ which give the direction of e^- in the ψ' rest frame.

We emphasize that the angles (θ',ϕ') of ψ and (θ'',ϕ'') of e^- are directions in the χ_J and the ψ rest frames, respectively. They are not the same as the corresponding angles measured in the ψ' rest frame or the lab frame. However, the different reference frames are related to each other through the Lorentz transformation. If $(\tilde{\theta}',\tilde{\phi}')$ represent the directions of ψ in the ψ' rest frame, they are related to the angles (θ',ϕ') by the following relations:

$$\phi' = \tilde{\phi}'\,, \tag{23}$$

$$\cos\theta' = \frac{1}{\left(1-\beta_2^2\cos^2\tilde{\theta}'\right)}\left\{(\cos^2\tilde{\theta}'-1)\frac{\beta_2}{\beta_1} + \cos\tilde{\theta}'\sqrt{1-\beta_2^2} \times\sqrt{1-\left(\frac{\beta_2}{\beta_1}\right)^2 + \cos^2\tilde{\theta}'\left[\left(\frac{\beta_2}{\beta_1}\right)^2 - \beta_2^2\right]}\right\}. \tag{24}$$

Since $0\le\theta'\le\pi$, $\sin\theta'$ has to be positive and so it will be given by the positive square root:

$$\sin\theta' = \sqrt{1-\cos^2\theta'}\,. \tag{25}$$

In (24) β_1 is the parameter v/c of ψ in the χ_J rest frame and β_2 is v/c of χ_J in the ψ' rest frame:

$$\beta_1 = \frac{M_{\chi_J}^2 - M_\psi^2}{M_{\chi_J}^2 + M_\psi^2}\,, \tag{26}$$

$$\beta_2 = \frac{M_{\psi'}^2 - M_{\chi_J}^2}{M_{\psi'}^2 + M_{\chi_J}^2}\,, \tag{27}$$

where $M_{\psi'}$ is the mass of the ψ' state. The angles (θ'',ϕ'') giving the directions of the electron in the ψ frame are related to the angles $(\tilde{\theta}'',\tilde{\phi}'')$ giving the directions of the electron in the ψ' frame by the relations:

$$\cos\phi'' = \frac{1}{\eta'}\left[\gamma_2\beta_2\sin\theta' + \cos\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'' + \cos\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'' - \gamma_2\sin\theta'\cos\tilde{\theta}''\right], \tag{28}$$

$$\sin\phi'' = \frac{1}{\eta'}\left[\cos\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'' - \sin\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''\right], \tag{29}$$

$$\begin{aligned}\cos\theta'' = \frac{1}{\eta}\big[&-\gamma_1\gamma_2(\beta_1+\beta_2\cos\theta') \\ &+\gamma_1(\sin\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'' \\ &+\sin\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'') \\ &+\gamma_1\gamma_2(\beta_1\beta_2+\cos\theta')\cos\tilde{\theta}''\big]\,,\end{aligned} \tag{30}$$

$$\sin\theta'' = +\sqrt{1-\cos^2\theta''} = \frac{\eta'}{\eta}\,, \tag{31}$$

where

$$\begin{aligned}\eta' = \big[&(\gamma_2\beta_2\sin\theta' + \cos\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'' \\ &+\cos\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'' - \gamma_2\sin\theta'\cos\tilde{\theta}'')^2 \\ &+(\cos\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'' - \sin\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'')^2\big]^{1/2}\,,\end{aligned} \tag{32}$$

$$\begin{aligned}\eta = \big[&\gamma_1\gamma_2(1+\beta_1\beta_2\cos\theta') - \gamma_1\beta_1(\sin\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'' \\ &+\sin\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'') \\ &-\gamma_1\gamma_2(\beta_2+\beta_1\cos\theta')\cos\tilde{\theta}''\big]\,.\end{aligned} \tag{33}$$

The constants γ_i $(i=1,2)$ are related to β_i $(i=1,2)$ by

$$\gamma_i = \frac{1}{\sqrt{1-\beta_i^2}}\,. \tag{34}$$

From (26) and (27)

$$\gamma_1 = \frac{M_{\chi_J}^2 + M_\psi^2}{2M_{\chi_J}M_\psi}\,, \tag{35}$$

$$\gamma_2 = \frac{M_{\psi'}^2 + M_{\chi_J}^2}{2M_{\psi'}M_{\chi_J}}\,. \tag{36}$$

Even though the above relations between the angles may look formidable, once the angular distribution is known in terms of the laboratory angles, they can easily be expressed in terms of the angles (θ,ϕ), (θ',ϕ') and (θ'',ϕ'') through a computer program generated with the help of these equations. This kind of transformation is routinely done by experimentalists.

Using (4), (6), (10) and (15) we can now write the amplitude in (1) as

$$T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2} = \frac{3\sqrt{3(2J+1)}}{(4\pi)^2}\sum_{\delta,\sigma}^{-1,0,+1}\sum_{\nu}^{-J\to+J} C_{\alpha_1\alpha_2}E^J_{\sigma\kappa}A^J_{\nu\mu} \times B_{\lambda_1\lambda_2}D^{1*}_{\sigma\alpha}(\phi'',\theta'',-\phi'') \times D^{J*}_{\nu,\sigma-\kappa}(\phi',\theta',-\phi')\delta_{\delta,\nu-\mu}D^1_{\delta\lambda}(\phi,\theta,-\phi)$$
$$= \frac{3\sqrt{3(2J+1)}}{(4\pi)^2}\sum_{\delta,\sigma}^{-1,0,+1} C_{\alpha_1\alpha_2}E^J_{\sigma\kappa}A^J_{\mu+\delta,\mu}B_{\lambda_1\lambda_2} \times D^{1*}_{\sigma\alpha}(\phi'',\theta'',-\phi'') \times D^{J*}_{\mu+\delta,\sigma-\kappa}(\phi',\theta',-\phi')D^1_{\delta\lambda}(\phi,\theta,-\phi)\,. \quad (37)$$

Because of the C and the P invariances [2], the angular-momentum helicity amplitudes in (37) are not all independent. We have

$$B_{\lambda_1\lambda_2} \overset{P}{=} B_{-\lambda_1,-\lambda_2} \overset{C}{=} B_{\lambda_2\lambda_1},$$
$$A^J_{\nu\mu} \overset{P}{=} (-1)^J A^J_{-\nu,-\mu},$$
$$E^J_{\sigma\kappa} \overset{P}{=} (-1)^J E^J_{-\sigma,-\kappa},$$

and

$$C_{\alpha_1\alpha_2} \overset{P}{=} C_{-\alpha_1,-\alpha_2} \overset{C}{=} C_{\alpha_2\alpha_1}\,. \quad (38)$$

Making use of the symmetry relations of (38) we now relabel the independent angular-momentum helicity amplitudes as follows:

$$B_\lambda = B_{(\lambda_1-\lambda_2)} = \sqrt{2}B_{\lambda_1\lambda_2} \qquad (\lambda = 0,1)\,,$$
$$A_\nu = A^J_{\nu,1} = (-1)^J A^J_{-\nu,-1} \qquad (\nu = 0,1,\ldots,+J)\,,$$
$$E_\sigma = E^J_{\sigma-1,-1} = (-1)^J E^J_{-\sigma+1,1} \qquad (\sigma = 0,1,\ldots,+J)\,,$$
$$C_\alpha = C_{(\alpha_1-\alpha_2)} = \sqrt{2}C_{\alpha_1\alpha_2} \qquad (\alpha = 0,1)\,. \quad (39)$$

When $\bar{p}$ and p are unpolarized, the normalized function describing the combined angular distribution of the electron and the two photons whose polarizations are also observed can be written as

$$W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = N_J\sum_{\lambda_1,\lambda_2}^{\pm\frac{1}{2}}\sum_{\alpha_2}^{\pm\frac{1}{2}} T^{\alpha_1\alpha_2\mu\kappa}_{\lambda_1\lambda_2}T^{\alpha_1\alpha_2\mu\kappa*}_{\lambda_1\lambda_2}\,, \quad (40)$$

where the subscripts $\mu\kappa\alpha_1$ of W represent the polarizations that are measured in the angular distribution. The normalization constant, N_J, in (40) is determined by requiring that the integral of the distribution function $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ over all the directions of γ_1, γ_2 and e^- or over all the angles, $(\theta,\phi;\theta',\phi';\theta'',\phi'')$, is 1. In (40) we sum over the helicities α_2 since e^+ is not observed. Substituting (37) into (40) and performing the various sums will then give us an expression for the angular distribution function $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ in terms of the Wigner D^J functions. After a very long algebra, we get

$$W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{3^2(2J+1)}{4(4\pi)^3}\sum_{L_1}^{0,1,2}\sum_{L_3}^{0,2}\varepsilon_{L_1}\alpha_{L_3} \times \sum_{L_2}^{0\to2J}(-1)^{\frac{1}{2}(1+\mu)L_2}(-1)^{\frac{1}{2}(1-\kappa)(L_1+L_2)} \times \sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\Big\{\beta^{L_3L_2}_{d+}\big[\gamma^{L_1L_2}_{d'+}(D_1+D_1^*+D_2+D_2^*) + \gamma^{L_1L_2}_{d'-}(D_1-D_1^*+D_2-D_2^*)\big] + \beta^{L_3L_2}_{d-}\big[\gamma^{L_1L_2}_{d'+}(D_1-D_1^*-D_2+D_2^*) + \gamma^{L_1L_2}_{d'-}(D_1+D_1^*-D_2-D_2^*)\big]\Big\}\,, \quad (41)$$

where

$$d_m = \min(L_3,L_2,J)\,,$$
$$d'_m = \min(L_1,L_2,J)\,, \quad (42)$$

and we have used the following normalizations for the angular-momentum helicity amplitudes B_λ, A_ν, E_σ and C_α defined in (39):

$$|B_0|^2+|B_1|^2 = |C_0|^2+|C_1|^2 = \sum_{\nu}^{0\to J}|A_\nu|^2 = \sum_{\sigma}^{0\to J}|E_\sigma|^2 = 1\,. \quad (43)$$

In (41) the angle-dependent terms are given by

$$D_1 = D^{L_3*}_{-\mu d,0}(\phi,\theta,-\phi)D^{L_1}_{\kappa d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{-\mu d,\kappa d'}(\phi',\theta',-\phi') \quad (44)$$

and

$$D_2 = D^{L_3*}_{\mu d,0}(\phi,\theta,-\phi)D^{L_1}_{\kappa d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{\mu d,\kappa d'}(\phi',\theta',-\phi')\,. \quad (45)$$

The coefficients ε_{L_1}, α_{L_3}, $\beta^{L_3L_2}_{d\pm}$ and $\gamma^{L_1L_2}_{d'\pm}$, which are independent of the angles in (41), are defined as follows:

$$\varepsilon_{L_1} = (-1)^{(\alpha_1-\frac{1}{2})L_1}\big\lfloor\langle 11;00\,|\,L_10\rangle|C_0|^2 - \langle 11;-11\,|\,L_10\rangle|C_1|^2\big\rfloor\,, \quad (46)$$

$$\alpha_{L_3} = \sum_{\lambda}^{0,1}(-1)^\lambda\langle 11;\lambda-\lambda\,|\,L_30\rangle|B_\lambda|^2\,, \quad (47)$$

$$\beta^{L_3L_2}_{d\pm} = \left(1-\frac{\delta_{d0}}{2}\right)\sum_{s(d)}\big[A_{\frac{s+d}{2}}A^*_{\frac{s-d}{2}} \pm A^*_{\frac{s+d}{2}}A_{\frac{s-d}{2}}\big] \times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\,\middle|\,L_2d\right\rangle \times\left\langle 11;\frac{s+d-2}{2},-\frac{s-d-2}{2}\,\middle|\,L_3d\right\rangle\,, \quad (48)$$

$$s(d) = |d|,|d|+2,\ldots,2J-|d|\,,$$
$$\gamma^{L_1L_2}_{d'\pm} = \beta^{L_1L_2}_{d'\pm} \quad (A\to E\,,\quad s(d)\to s'(d'))\,. \quad (49)$$

The explicit expressions for the nonzero coefficients $\gamma_{d'\pm}^{L_1L_2}$ as well as ε_{L_1} and α_{L_3} are given in the following; we have

$$\alpha_0 = -\frac{1}{\sqrt{3}}\left[|B_0|^2+|B_1|^2\right] = -\frac{1}{\sqrt{3}}\,,$$
$$\alpha_2 = \frac{1}{\sqrt{6}}\left[2|B_0|^2-|B_1|^2\right]\,, \tag{50}$$

$$\varepsilon_0 = -\frac{1}{\sqrt{3}}\left[|C_0|^2+|C_1|^2\right] = -\frac{1}{\sqrt{3}}\,,$$
$$\varepsilon_1 = \frac{1}{\sqrt{2}}(-1)^{\left(\alpha_1-\frac{1}{2}\right)}|C_1|^2 \cong \frac{1}{\sqrt{2}}(-1)^{\left(\alpha_1-\frac{1}{2}\right)}\,,$$
$$\varepsilon_2 = \frac{1}{\sqrt{6}}\left[2|C_0|^2-|C_1|^2\right] \cong -\frac{1}{\sqrt{6}}\,. \tag{51}$$

$J=0$:

$$\gamma_{0+}^{00} = \frac{1}{\sqrt{3}}\,,$$
$$\gamma_{0+}^{10} = -\frac{1}{\sqrt{2}}\,,$$
$$\gamma_{0+}^{20} = \frac{1}{\sqrt{6}}\,. \tag{52}$$

$J=1$:

$$\gamma_{0+}^{00} = -\frac{1}{3}\,,$$
$$\gamma_{0+}^{01} = -\frac{1}{\sqrt{6}}|E_1|^2\,,$$
$$\gamma_{0+}^{02} = \frac{1}{3\sqrt{2}}\left(2|E_0|^2-|E_1|^2\right)\,,$$
$$\gamma_{0+}^{10} = \frac{1}{\sqrt{6}}|E_0|^2\,,$$
$$\gamma_{1+}^{11} = -\,\mathrm{Re}(E_1E_0^*)\,,$$
$$\gamma_{0+}^{12} = -\frac{1}{\sqrt{3}}|E_0|^2\,,$$
$$\gamma_{1+}^{12} = -\,\mathrm{Re}(E_1E_0^*)\,,$$
$$\gamma_{0+}^{20} = -\frac{1}{3\sqrt{2}}\left(|E_0|^2-2|E_1|^2\right)\,,$$
$$\gamma_{0+}^{21} = \frac{1}{\sqrt{3}}|E_1|^2\,,$$
$$\gamma_{1+}^{21} = \mathrm{Re}(E_1E_0^*)\,,$$
$$\gamma_{0+}^{22} = \frac{1}{3}\,,$$
$$\gamma_{1+}^{22} = \mathrm{Re}(E_1E_0^*)\,,$$
$$\gamma_{1-}^{11} = -\mathrm{i}\,\mathrm{Im}(E_1E_0^*)\,,$$
$$\gamma_{1-}^{12} = -\mathrm{i}\,\mathrm{Im}(E_1E_0^*)\,,$$
$$\gamma_{1-}^{21} = \mathrm{i}\,\mathrm{Im}(E_1E_0^*)\,,$$
$$\gamma_{1-}^{22} = \mathrm{i}\,\mathrm{Im}(E_1E_0^*)\,. \tag{53}$$

$J=2$:

$$\gamma_{0+}^{00} = \frac{1}{\sqrt{15}}\,,$$
$$\gamma_{0+}^{01} = \frac{1}{\sqrt{30}}\left(|E_1|^2+2|E_2|^2\right)\,,$$
$$\gamma_{0+}^{02} = -\frac{1}{\sqrt{42}}\left(2|E_0|^2+|E_1|^2-2|E_2|^2\right)\,,$$
$$\gamma_{0+}^{03} = -\frac{1}{\sqrt{30}}\left(2|E_1|^2-|E_2|^2\right)\,,$$
$$\gamma_{0+}^{04} = \frac{1}{\sqrt{210}}\left(6|E_0|^2-4|E_1|^2+|E_2|^2\right)\,,$$
$$\gamma_{0+}^{10} = -\frac{1}{\sqrt{10}}\left(|E_0|^2-|E_2|^2\right)\,,$$
$$\gamma_{0+}^{11} = \frac{1}{\sqrt{5}}|E_2|^2\,,$$
$$\gamma_{1+}^{11} = \frac{1}{\sqrt{5}}\left[\sqrt{3}\,\mathrm{Re}(E_1E_0^*)+\sqrt{2}\,\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{0+}^{12} = \frac{1}{\sqrt{7}}\left(|E_0|^2+|E_2|^2\right)\,,$$
$$\gamma_{1+}^{12} = \frac{1}{\sqrt{7}}\left[\mathrm{Re}(E_1E_0^*)+\sqrt{6}\,\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{0+}^{13} = \frac{1}{2\sqrt{5}}|E_2|^2\,,$$
$$\gamma_{1+}^{13} = -\frac{1}{\sqrt{5}}\left[\sqrt{2}\,\mathrm{Re}(E_1E_0^*)-\sqrt{3}\,\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{0+}^{14} = -\frac{1}{2\sqrt{35}}\left(6|E_0|^2-|E_2|^2\right)\,,$$
$$\gamma_{1+}^{14} = -\frac{1}{\sqrt{7}}\left[\sqrt{6}\,\mathrm{Re}(E_1E_0^*)-\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{0+}^{20} = \frac{1}{\sqrt{30}}\left(|E_0|^2-2|E_1|^2+|E_2|^2\right)\,,$$
$$\gamma_{0+}^{21} = -\frac{1}{\sqrt{15}}\left(|E_1|^2-|E_2|^2\right)\,,$$
$$\gamma_{1+}^{21} = -\frac{1}{\sqrt{5}}\left[\sqrt{3}\,\mathrm{Re}(E_1E_0^*)-\sqrt{2}\,\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{0+}^{22} = -\frac{1}{\sqrt{21}}\left(|E_0|^2-|E_1|^2-|E_2|^2\right)\,,$$
$$\gamma_{1+}^{22} = -\frac{1}{\sqrt{7}}[\mathrm{Re}(E_1E_0^*)-\sqrt{6}\,\mathrm{Re}(E_2E_1^*)]\,,$$
$$\gamma_{2+}^{22} = \sqrt{\frac{8}{7}}\,\mathrm{Re}(E_2E_0^*)\,,$$
$$\gamma_{0+}^{23} = \frac{1}{2\sqrt{15}}\left(4|E_1|^2+|E_2|^2\right)\,,$$
$$\gamma_{1+}^{23} = \frac{1}{\sqrt{5}}\left[\sqrt{2}\,\mathrm{Re}(E_1E_0^*)+\sqrt{3}\,\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{2+}^{23} = \sqrt{2}\,\mathrm{Re}(E_2E_0^*)\,,$$
$$\gamma_{0+}^{24} = \frac{1}{\sqrt{420}}\left(6|E_0|^2+8|E_1|^2+|E_2|^2\right)\,,$$
$$\gamma_{1+}^{24} = \frac{1}{\sqrt{7}}\left[\sqrt{6}\,\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*)\right]\,,$$
$$\gamma_{2+}^{24} = \sqrt{\frac{6}{7}}\,\mathrm{Re}(E_2E_0^*)\,,$$
$$\gamma_{1-}^{11} = \frac{\mathrm{i}}{\sqrt{5}}\left[\sqrt{3}\,\mathrm{Im}(E_1E_0^*)+\sqrt{2}\,\mathrm{Im}(E_2E_1^*)\right]\,,$$
$$\gamma_{1-}^{12} = \frac{\mathrm{i}}{\sqrt{7}}\left[\mathrm{Im}(E_1E_0^*)+\sqrt{6}\,\mathrm{Im}(E_2E_1^*)\right]\,,$$
$$\gamma_{1-}^{13} = -\frac{\mathrm{i}}{\sqrt{5}}\left[\sqrt{2}\,\mathrm{Im}(E_1E_0^*)-\sqrt{3}\,\mathrm{Im}(E_2E_1^*)\right]\,,$$

$$\gamma_{1-}^{14} = -\frac{\mathrm{i}}{\sqrt{7}}\left[\sqrt{6}\,\mathrm{Im}(E_1E_0^*) - \mathrm{Im}(E_2E_1^*)\right],$$
$$\gamma_{1-}^{21} = -\frac{\mathrm{i}}{\sqrt{5}}\left[\sqrt{3}\,\mathrm{Im}(E_1E_0^*) - \sqrt{2}\,\mathrm{Im}(E_2E_1^*)\right],$$
$$\gamma_{1-}^{22} = -\frac{\mathrm{i}}{\sqrt{7}}\left[\mathrm{Im}(E_1E_0^*) - \sqrt{6}\,\mathrm{Im}(E_2E_1^*)\right],$$
$$\gamma_{2-}^{22} = \mathrm{i}\sqrt{\frac{8}{7}}\,\mathrm{Im}(E_2E_0^*),$$
$$\gamma_{1-}^{23} = \frac{\mathrm{i}}{\sqrt{5}}\left[\sqrt{2}\,\mathrm{Im}(E_1E_0^*) + \sqrt{3}\,\mathrm{Im}(E_2E_1^*)\right],$$
$$\gamma_{2-}^{23} = \mathrm{i}\sqrt{2}\,\mathrm{Im}(E_2E_0^*),$$
$$\gamma_{1-}^{24} = \frac{\mathrm{i}}{\sqrt{7}}\left[\sqrt{6}\,\mathrm{Im}(E_1E_0^*) + \mathrm{Im}(E_2E_1^*)\right],$$
$$\gamma_{2-}^{24} = \mathrm{i}\sqrt{\frac{6}{7}}\,\mathrm{Im}(E_2E_0^*). \quad (54)$$

The expressions for $\beta_{d\pm}^{L_3L_2}$ are identical to $\gamma_{d\pm}^{L_3L_2}$, provided that we replace the helicity amplitudes E by A. In calculating (50)–(54) we assumed the normalization conditions in (43). Note that we have more $\gamma_{d'\pm}^{L_1L_2}$ than $\beta_{d\pm}^{L_3L_2}$ since L_1 takes the values 0, 1 and 2 while L_3 only takes 0 and 2.

Since the combined angular distribution in (41) is expressed as a sum of products of the orthogonal Wigner D^J functions, we can obtain the coefficients of the D^J angular functions as

$$\begin{aligned}
&3^2(2J+1)\varepsilon_{L_1}\alpha_{L_3}(-1)^{\frac{1}{2}(1+\mu)L_2}(-1)^{\frac{1}{2}(1-\kappa)(L_1+L_2)}\\
&\quad\times\big\{\beta_{d+}^{L_3L_2}\big[\gamma_{d'+}^{L_1L_2}(1+\delta_{d0})(1+\delta_{d'0})\\
&\quad+\gamma_{d'-}^{L_1L_2}(1+\delta_{d0})(1-\delta_{d'0})\big]\\
&\quad+\beta_{d-}^{L_3L_2}\big[\gamma_{d'+}^{L_1L_2}(1-\delta_{d0})(1+\delta_{d'0})\\
&\quad+\gamma_{d'-}^{L_1L_2}(1-\delta_{d0})(1-\delta_{d'0})\big]\big\}\\
&=4(2L_1+1)(2L_2+1)(2L_3+1)\\
&\quad\times\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')D_1^*\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}\Omega''\,.
\end{aligned} \quad (55)$$

In calculating (55), we made use of the orthogonality relation,

$$\int_0^{2\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\gamma\int_0^{\pi}D_{mm'}^{j*}(\alpha,\beta,\gamma)D_{\mu\mu'}^{j'}(\alpha,\beta,\gamma)\sin\beta\,\mathrm{d}\beta = \frac{8\pi^2}{(2j+1)}\delta_{m\mu}\delta_{m'\mu'}\delta_{jj'}\,. \quad (56)$$

When we have sufficient experimental data for the angular distribution function $W_{\mu\kappa\alpha_1}$ where the final polarizations, μ, κ and α_1, of all the three decay particles are measured, the integral on the right side of (55) can then be determined numerically for all possible allowed values of L_1, L_2, L_3, d and d'. Thus we can obtain the different coefficients ε_{L_1}, α_{L_3}, $\beta_{d\pm}^{L_3L_2}$ and $\gamma_{d'\pm}^{L_1L_2}$ on the left side of (55). From these coefficients we can determine the relative magnitudes as well as the relative phases of the A and the E helicity amplitudes in the radiative decay processes $\psi'\to\chi_J+\gamma_1$ and $\chi_J\to\psi+\gamma_2$, respectively, for the $J=1$ and the $J=2$ cases. For the $J=0$ case, there is only one independent helicity amplitude for each of the radiative decay process and that is fixed by our normalization. We can also obtain the relative magnitudes of the B helicity amplitudes in the initial process $\bar{p}p\to\psi'$ and the C helicity amplitude in the final decay process $\psi\to e^+e^-$ for all values of J. For example, in the $J=1$ case, the measurements of the $(L_1L_2L_3dd')$ coefficients will give us the following. First the measurement of the (10000) and the (12200) coefficients yields α_2, which, with the normalization $|B_0|^2+|B_1|^2=1$, enables us to determine the relative magnitudes of B_0 and B_1. Next measuring the (00200) coefficient gives β_{0+}^{20}, and with the normalization $|A_0|^2+|A_1|^2=1$, the relative magnitudes of A_0 and A_1 are also determined. Then measuring the (01000) coefficient gives γ_{0+}^{01} and hence $|E_1|^2$. With the normalization $|E_0|^2+|E_1|^2=1$ we can determine the relative magnitudes of E_0 and E_1. The relative magnitudes of C_0 and C_1 can then be obtained from the measurement of the (10000) coefficient and the normalization $|C_0|^2+|C_1|^2=1$. After having obtained all the relative magnitudes, now measuring the (12001) coefficient gives both $\mathrm{Re}(E_1E_0^*)$ and $\mathrm{Im}(E_1E_0^*)$. Thus the relative phase between E_0 and E_1 is determined. Finally the measurement of the (02210) coefficient yields both $\mathrm{Re}(A_1A_0^*)$ and $\mathrm{Im}(A_1A_0^*)$. Hence the relative phase between A_0 and A_1 is also obtained.

It is interesting to note that using (41) we can easily obtain different combined angular distribution functions where the polarizations of only one or two of the decay products γ_1, γ_2 and e^- are measured. Suppose we are interested in only measuring the polarization μ of γ_1, the normalized combined angular distribution of γ_1, γ_2 and e^- will then become

$$\begin{aligned}
&W_\mu(\theta,\phi;\theta',\phi';\theta'',\phi'')\\
&=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\kappa}^{\pm 1}W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\\
&=\frac{3^2(2J+1)}{8(4\pi)^3}\sum_{L_1}^{0,2}\sum_{L_3}^{0,2}\varepsilon_{L_1}\alpha_{L_3}\\
&\times\sum_{L_2}^{0\to 2J}(-1)^{\frac{1}{2}(1+\mu)L_2}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\big\{\beta_{d+}^{L_3L_2}\\
&\times\big[\gamma_{d'+}^{L_1L_2}(1+(-1)^{L_2})(\tilde{D}_1+\tilde{D}_1^*+\tilde{D}_2+\tilde{D}_2^*)\\
&+\gamma_{d'-}^{L_1L_2}(1-(-1)^{L_2})(\tilde{D}_1-\tilde{D}_1^*+\tilde{D}_2-\tilde{D}_2^*)\big]\\
&+\beta_{d-}^{L_3L_2}\big[\gamma_{d'+}^{L_1L_2}(1+(-1)^{L_2})(\tilde{D}_1-\tilde{D}_1^*-\tilde{D}_2+\tilde{D}_2^*)\\
&+\gamma_{d'-}^{L_1L_2}(1-(-1)^{L_2})(\tilde{D}_1+\tilde{D}_1^*-\tilde{D}_2-\tilde{D}_2^*)\big]\big\}\,,
\end{aligned} \quad (57)$$

where

$$\begin{aligned}\tilde{D}_1 = D_1(\kappa=1) &= D_{-\mu d,0}^{L_3*}(\phi,\theta,-\phi)D_{d',0}^{L_1}(\phi'',\theta'',-\phi'')\\ &\times D_{-\mu d,d'}^{L_2}(\phi',\theta',-\phi')\end{aligned} \quad (58)$$

and

$$\tilde{D}_2 = D_2(\kappa=1) = D^{L_3*}_{\mu d,0}(\phi,\theta,-\phi) D^{L_1}_{d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{\mu d,d'}(\phi',\theta',-\phi') \,. \quad (59)$$

The coefficients of the D^J angular functions in (57) can be obtained from

$$\begin{aligned} &3^2(2J+1)\varepsilon_{L_1}\alpha_{L_3}(-1)^{\frac{1}{2}(1+\mu)L_2} \\ &\times \{\beta^{L_3L_2}_{d+}[\gamma^{L_1L_2}_{d'+}(1+(-1)^{L_2})(1+\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}(1-(-1)^{L_2})(1+\delta_{d0})(1-\delta_{d'0})] \\ &+\beta^{L_3L_2}_{d-}[\gamma^{L_1L_2}_{d'+}(1+(-1)^{L_2})(1-\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}(1-(-1)^{L_2})(1-\delta_{d0})(1-\delta_{d'0})]\} \\ &= 8(2L_1+1)(2L_2+1)(2L_3+1) \\ &\times \int W_\mu(\theta,\phi;\theta',\phi';\theta'',\phi'')\tilde{D}^*_1 \,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}\Omega'' \,, \end{aligned} \quad (60)$$

where L_1 can only take the values 0 and 2.

Similarly, the normalized combined angular distribution of γ_1, γ_2 and e^- where only the polarization κ of γ_2 is measured can be written as

$$\begin{aligned} &W_\kappa(\theta,\phi;\theta',\phi';\theta'',\phi'') \\ &= \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm 1} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\ &= \frac{3^2(2J+1)}{8(4\pi)^3}\sum_{L_1}^{0,2}\sum_{L_3}^{0,2}\varepsilon_{L_1}\alpha_{L_3} \\ &\times \sum_{L_2}^{0\to 2J}(-1)^{\frac{1}{2}(1-\kappa)L_2}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\{\beta^{L_3L_2}_{d+} \\ &\times [\gamma^{L_1L_2}_{d'+}((-1)^{L_2}+1)(\hat{D}_1+\hat{D}^*_1+\hat{D}_2+\hat{D}^*_2) \\ &+\gamma^{L_1L_2}_{d'-}((-1)^{L_2}+1)(\hat{D}_1-\hat{D}^*_1+\hat{D}_2-\hat{D}^*_2)] \\ &+\beta^{L_3L_2}_{d-}[\gamma^{L_1L_2}_{d'+}((-1)^{L_2}-1)(\hat{D}_1-\hat{D}^*_1-\hat{D}_2+\hat{D}^*_2) \\ &+\gamma^{L_1L_2}_{d'-}((-1)^{L_2}-1)(\hat{D}_1+\hat{D}^*_1-\hat{D}_2-\hat{D}^*_2)]\} \,, \end{aligned} \quad (61)$$

where

$$\hat{D}_1 = D_1(\mu=1) = D^{L_3*}_{-d,0}(\phi,\theta,-\phi) D^{L_1}_{\kappa d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{-d,\kappa d'}(\phi',\theta',-\phi') \quad (62)$$

and

$$\hat{D}_2 = D_2(\mu=1) = D^{L_3*}_{d,0}(\phi,\theta,-\phi) D^{L_1}_{\kappa d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{d,\kappa d'}(\phi',\theta',-\phi') \,. \quad (63)$$

The coefficients in (61) can be obtained as

$$\begin{aligned} &3^2(2J+1)\varepsilon_{L_1}\alpha_{L_3}(-1)^{\frac{1}{2}(1-\kappa)L_2} \\ &\times \{\beta^{L_3L_2}_{d+}[\gamma^{L_1L_2}_{d'+}((-1)^{L_2}+1)(1+\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}((-1)^{L_2}+1)(1+\delta_{d0})(1-\delta_{d'0})] \\ &+\beta^{L_3L_2}_{d-}[\gamma^{L_1L_2}_{d'+}((-1)^{L_2}-1)(1-\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}((-1)^{L_2}-1)(1-\delta_{d0})(1-\delta_{d'0})]\} \\ &= 8(2L_1+1)(2L_2+1)(2L_3+1) \\ &\times \int W_\kappa(\theta,\phi;\theta',\phi';\theta'',\phi'')\hat{D}^*_1 \,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}\Omega'' \,, \end{aligned} \quad (64)$$

where again L_1 can only take the values 0 and 2.

If we are only interested in measuring the polarization α_1 of e^-, the combined angular distribution of γ_1, γ_2 and e^- will become

$$\begin{aligned} &W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\ &= \frac{1}{4}\sum_{\kappa}^{\pm 1}\sum_{\mu}^{\pm 1} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\ &= \frac{3^2(2J+1)}{16(4\pi)^3}\sum_{L_1}^{0,1,2}\sum_{L_3}^{0,2}\varepsilon_{L_1}\alpha_{L_3} \\ &\times \sum_{L_2}^{0\to 2J}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\{\beta^{L_3L_2}_{d+}((-1)^{L_2}+1) \\ &\times [\gamma^{L_1L_2}_{d'+}(1+(-1)^{L_1})(D'_1+D'^*_1+D'_2+D'^*_2) \\ &+\gamma^{L_1L_2}_{d'-}(1-(-1)^{L_1})(D'_1-D'^*_1+D'_2-D'^*_2)] \\ &+\beta^{L_3L_2}_{d-}((-1)^{L_2}-1) \\ &\times [\gamma^{L_1L_2}_{d'+}(1-(-1)^{L_1})(D'_1-D'^*_1-D'_2+D'^*_2) \\ &+\gamma^{L_1L_2}_{d'-}(1+(-1)^{L_1})(D'_1+D'^*_1-D'_2-D'^*_2)]\} \,, \end{aligned} \quad (65)$$

where

$$D'_1 = D_1(\mu=\kappa=1) = D^{L_3*}_{-d,0}(\phi,\theta,-\phi) D^{L_1}_{d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{-d,d'}(\phi',\theta',-\phi') \quad (66)$$

and

$$D'_2 = D_2(\mu=\kappa=1) = D^{L_3*}_{d,0}(\phi,\theta,-\phi) D^{L_1}_{d',0}(\phi'',\theta'',-\phi'') \times D^{L_2}_{d,d'}(\phi',\theta',-\phi') \,. \quad (67)$$

The coefficients in (65) can be obtained as

$$\begin{aligned} &3^2(2J+1)\varepsilon_{L_1}\alpha_{L_3}\{\beta^{L_3L_2}_{d+}((-1)^{L_2}+1) \\ &\times [\gamma^{L_1L_2}_{d'+}(1+(-1)^{L_1})(1+\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}(1-(-1)^{L_1})(1+\delta_{d0})(1-\delta_{d'0})] \\ &+\beta^{L_3L_2}_{d-}((-1)^{L_2}-1) \\ &\times [\gamma^{L_1L_2}_{d'+}(1-(-1)^{L_1})(1-\delta_{d0})(1+\delta_{d'0}) \\ &+\gamma^{L_1L_2}_{d'-}(1+(-1)^{L_1})(1-\delta_{d0})(1-\delta_{d'0})]\} \\ &= 16(2L_1+1)(2L_2+1)(2L_3+1) \\ &\times \int W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')D'^*_1 \,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}\Omega'' \,, \end{aligned} \quad (68)$$

where this time L_1 can take the values 0, 1 and 2. If we now average over the polarizations α_1 of e^- in (65) as well, we get

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha_1}^{\pm\frac{1}{2}} W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\
&= \frac{1}{8}\sum_{\kappa}^{\pm1}\sum_{\mu}^{\pm1}\sum_{\alpha_1}^{\pm\frac{1}{2}} W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'') \\
&= \frac{3^2(2J+1)}{8(4\pi)^3}\sum_{L_1}^{0,2}\sum_{L_3}^{0,2}\varepsilon_{L_1}\alpha_{L_3} \\
&\times \sum_{L_2}^{0\to 2J}\sum_{d}^{0\to d_m}\sum_{d'}^{0\to d'_m}\left\{\left[\beta_{d+}^{L_3L_2}\gamma_{d'+}^{L_1L_2}+\beta_{d-}^{L_3L_2}\gamma_{d'-}^{L_1L_2}\right]\right. \\
&\left. +(-1)^{L_2}\left[\beta_{d+}^{L_3L_2}\gamma_{d'+}^{L_1L_2}-\beta_{d-}^{L_3L_2}\gamma_{d'-}^{L_1L_2}\right]\right\} \\
&\times\left[(D'_2+D'^*_2)+(-1)^{L_2}(D'_1+D'^*_1)\right]. \qquad (69)
\end{aligned}$$

Using (48) and (49), the terms inside the braces of (69) can be simplified as

$$\begin{aligned}
&\left[\beta_{d+}^{L_3L_2}\gamma_{d'+}^{L_1L_2}+\beta_{d-}^{L_3L_2}\gamma_{d'-}^{L_1L_2}\right] \\
&\quad +(-1)^{L_2}\left[\beta_{d+}^{L_3L_2}\gamma_{d'+}^{L_1L_2}-\beta_{d-}^{L_3L_2}\gamma_{d'-}^{L_1L_2}\right] \\
&= \beta_{d}^{L_3L_2}\gamma_{d'}^{L_1L_2}, \qquad (70)
\end{aligned}$$

where

$$\begin{aligned}
\beta_{d}^{L_3L_2} &= \left(1-\frac{\delta_{d0}}{2}\right)\sum_{s(d)}\left[A_{\frac{s+d}{2}}A^*_{\frac{s-d}{2}}+(-1)^{L_2}A^*_{\frac{s+d}{2}}A_{\frac{s-d}{2}}\right] \\
&\times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\,\middle|\,L_2 d\right\rangle \\
&\times\left\langle 11;\frac{s+d-2}{2},-\frac{s-d-2}{2}\,\middle|\,L_3 d\right\rangle, \qquad (71)
\end{aligned}$$

$$\begin{aligned}
&s(d) = |d|, |d|+2, \ldots, 2J-|d|, \\
&\gamma_{d'}^{L_1L_2} = \beta_{d'}^{L_1L_2} \quad (A\to E \text{ and } s(d)\to s'(d')). \qquad (72)
\end{aligned}$$

By combining (69) and (70), we now recover the results in [5], where the polarizations of the decay particles are not measured.

Using (60), (64) or (68) it can be seen that once the combined angular distribution $W_\mu(\theta,\phi;\theta',\phi';\theta'',\phi'')$, $W_\kappa(\theta,\phi;\theta',\phi';\theta'',\phi'')$ or $W_{\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ is measured, one can also get the same information on the helicity amplitudes as one obtained from measuring $W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')$ where the polarizations of the three particles γ_1, γ_2 and e^- are observed. In other words, by measuring the combined angular distribution of the decay particles γ_1, γ_2 and e^- and the polarization of any one particle, we can get complete information on the helicity amplitudes in the radiative decay processes $\psi'\to\chi_J+\gamma_1$ and $\chi_J\to\psi+\gamma_2$. In addition, we can also get the relative magnitudes of the helicity amplitudes in the production process $\bar{p}p\to\psi'$ and in the final decay process $\psi\to e^+e^-$.

3 Partially integrated angular distributions

The partially integrated angular distributions obtained from (41) will look a lot simpler and we shall gain greater insight from them. We now consider six different cases of partially integrated angular distributions. In deriving these results, we shall frequently make use of (56) and the following property of the D^J functions:

$$\begin{aligned}
&\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}D_{MM'}^{L*}(\phi,\theta,-\phi)\sin\theta\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}D_{MM'}^{L}(\phi,\theta,-\phi)\sin\theta\,\mathrm{d}\theta \\
&= 2\pi\delta_{M-M',0}\int_0^{\pi}d_{MM'}^{L}(\theta)\sin\theta\,\mathrm{d}\theta = 2\pi k_{LM}, \qquad (73)
\end{aligned}$$

where

$$k_{LM} = \int_0^{\pi}d_{MM}^{L}(\theta)\sin\theta\,\mathrm{d}\theta. \qquad (74)$$

We will express the final results in terms of the orthogonal spherical harmonics by making use of the relation:

$$D_{M0}^{L} = \sqrt{\frac{4\pi}{2L+1}}Y_{LM}^*. \qquad (75)$$

Case 1: we shall integrate over (θ',ϕ') and (θ'',ϕ'') and average over the polarizations for γ_2 and e^-. Only the polarization and the angular distribution of the first gamma photon γ_1 are measured. We obtain

$$\begin{aligned}
\tilde{W}_\mu(\theta,\phi) &= \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\kappa}^{\pm1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\,\mathrm{d}\Omega'\,\mathrm{d}\Omega'' \\
&= \frac{9(2J+1)}{2\sqrt{\pi}}\varepsilon_0\gamma_{0+}^{00} \\
&\times\left[\alpha_0\beta_{0+}^{00}Y_{00}(\theta)+\frac{1}{\sqrt{5}}\alpha_2\beta_{0+}^{20}Y_{20}(\theta)\right], \qquad (76)
\end{aligned}$$

where the angle θ is the direction of $\bar{p}$ measured from the z axis, which is taken to be the direction of the momentum of χ_J. This angle is the same as that of γ_1 measured in the ψ' rest frame with the z axis taken to be the direction of the proton. The x and y axes are arbitrary in our discussion. Note that in (76) $\tilde{W}_\mu(\theta,\phi)$ is independent of the polarization μ. In other words, the single-particle angular distribution of γ_1 is the same whether the polarization of γ_1 is measured or not. Substituting for the coefficients α, ε, γ and β given in (50)–(54), we can express $\tilde{W}_\mu(\theta,\phi)$ in terms of the helicity amplitudes for the $J=0$, $J=1$ and $J=2$ cases as follows.

$J=0$:

$$\tilde{W}_\mu(\theta,\phi) = \frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta)-\frac{1}{\sqrt{5}}\left(2|B_0|^2-|B_1|^2\right)Y_{20}(\theta)\right]. \qquad (77)$$

$J=1$:

$$\tilde{W}_\mu(\theta,\phi)=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta)-\frac{1}{\sqrt{5}}\left(2|B_0|^2-|B_1|^2\right)\times\left(|A_0|^2-2|A_1|^2\right)Y_{20}(\theta)\right]. \qquad (78)$$

$J=2$:

$$\tilde{W}_\mu(\theta,\phi)=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta)-\frac{1}{\sqrt{5}}\left(2|B_0|^2-|B_1|^2\right)\times\left(|A_0|^2-2|A_1|^2+|A_2|^2\right)Y_{20}(\theta)\right]. \qquad (79)$$

Case 2: we shall integrate over (θ,ϕ) and (θ'',ϕ'') and average over the polarizations for γ_1 and e^-. Only the polarization and the angular distribution of the second gamma photon γ_2 are measured. We get

$$\tilde{W}_\kappa(\theta',\phi')=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\,\mathrm{d}\Omega\,\mathrm{d}\Omega''$$
$$=\frac{9(2J+1)}{2\sqrt{\pi}}\varepsilon_0\alpha_0\sum_{L_2}^{0,2\to2J}\beta_{0+}^{0L_2}\gamma_{0+}^{0L_2}\times\sqrt{\frac{1}{2L_2+1}}Y_{L_20}(\theta'), \qquad (80)$$

where θ' is the angle between ψ' and γ_2 in the χ_J rest frame. Note also that in (80) $\tilde{W}_\kappa(\theta',\phi')$ is independent of the polarization of γ_2. Therefore, the photon γ_2 with positive helicity has the same single-particle angular distribution as the photon γ_2 with negative helicity. Using (50)–(54) we again write the results separately for the $J=0$, $J=1$ and $J=2$ cases in terms of the helicity amplitudes.

$J=0$:

$$\tilde{W}_\kappa(\theta',\phi')=\frac{1}{2\sqrt{\pi}}Y_{00}(\theta'), \qquad (81)$$

$J=1$:

$$\tilde{W}_\kappa(\theta',\phi')=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta')+\frac{1}{2\sqrt{5}}\left(2|A_0|^2-|A_1|^2\right)\times\left(2|E_0|^2-|E_1|^2\right)Y_{20}(\theta')\right]. \qquad (82)$$

$J=2$:

$$\tilde{W}_\kappa(\theta',\phi')=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta')+\frac{\sqrt{5}}{14}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\times\left(2|E_0|^2+|E_1|^2-2|E_2|^2\right)Y_{20}(\theta')+\frac{1}{42}\left(6|A_0|^2-4|A_1|^2+|A_2|^2\right)\times\left(6|E_0|^2-4|E_1|^2+|E_2|^2\right)Y_{40}(\theta')\right]. \qquad (83)$$

Case 3: we shall integrate over (θ,ϕ) and (θ',ϕ') and average over the polarizations for γ_1 and γ_2. Only the polarization and the angular distribution of the electron are measured. We have

$$\tilde{W}_{\alpha_1}(\theta'',\phi'')=\frac{1}{4}\sum_{\mu}^{\pm1}\sum_{\kappa}^{\pm1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\,\mathrm{d}\Omega\,\mathrm{d}\Omega'$$
$$=\frac{9(2J+1)}{2\sqrt{\pi}}\alpha_0\beta_{0+}^{00}\sum_{L_1}^{0,2}\varepsilon_{L_1}\gamma_{0+}^{L_10}\sqrt{\frac{1}{2L_1+1}}Y_{L_10}(\theta''), \qquad (84)$$

where θ'' is the "Wigner-rotated" angle between the directions of the momenta of e^- and χ_J in the ψ rest frame. Note that in (84) $\tilde{W}_{\alpha_1}(\theta'',\phi'')$ is independent of the helicity α_1 of the electron. The following expressions are the results for the $J=0$, $J=1$ and $J=2$ cases, respectively, in terms of the helicity amplitudes:

$J=0$:

$$\tilde{W}_{\alpha_1}(\theta'',\phi'')=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta'')+\frac{1}{2\sqrt{5}}Y_{20}(\theta'')\right]. \qquad (85)$$

$J=1$:

$$\tilde{W}_{\alpha_1}(\theta'',\phi'')=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta'')+\frac{1}{2\sqrt{5}}\left(|E_0|^2-2|E_1|^2\right)Y_{20}(\theta'')\right]. \qquad (86)$$

$J=2$:

$$\tilde{W}_{\alpha_1}(\theta'',\phi'')=\frac{1}{2\sqrt{\pi}}\left[Y_{00}(\theta'')+\frac{1}{2\sqrt{5}}\left(|E_0|^2-2|E_1|^2+|E_2|^2\right)Y_{20}(\theta'')\right]. \qquad (87)$$

In calculating (85)–(87) we have neglected $|C_0|^2$ when compared to $|C_1|^2$ for the reason mentioned earlier. Using (86) and the normalization $|E_0|^2+|E_1|^2=1$, we can easily express the relative magnitudes of the E helicity amplitudes for $J=1$ in terms of the angular distribution $\tilde{W}_{\alpha_1}(\theta'',\phi'')$. Therefore the measurement of the angular distribution of the electron alone enables us to determine the relative magnitudes of the E helicity amplitudes. Likewise, we can then get the relative magnitudes of the A helicity amplitudes for $J=1$ by (82) together with the normalization $|A_0|^2+|A_1|^2=1$. Finally we can determine the relative magnitudes of the B helicity amplitudes for $J=1$ by making use of (78) and the normalization $|B_0|^2+|B_1|^2=1$. In other words, we can obtain the relative magnitudes of all the helicity amplitudes B, A and E for the $J=1$ case by measuring the one-particle angular distributions of the electron and the two photons. For the $J=2$ case, this is not possible because there are more unknowns than the equations. Using (77) we can also determine the relative magnitudes of the B helicity amplitudes for the $J=0$ case. This will give us a check on what we had in the $J=1$ case because the production amplitudes in the initial process $\bar{p}p \to \psi'$ should be independent of J.

After studying the single-particle angular distributions for γ_1, γ_2 and e^- whose polarizations are also measured, we can conclude that these angular distributions are the same as the single-particle angular distributions with no measurement of polarizations. Therefore the measurement of the polarizations of the particles does not give us any extra information. Nevertheless, we will find that the observation of the polarizations of the decay particles is useful and gives us more information on the helicity amplitudes when we measure the simultaneous angular distributions of two particles. Since there is nothing more to be determined from the $J=0$ case, we shall only concentrate on the $J=1$ and the $J=2$ cases.

Case 4: we shall first integrate over the angles (θ,ϕ) and then average over the polarizations of γ_1. The combined angular distribution of γ_2 and e^- and the polarization of only one of the two particles are measured. The explicit expressions are given in the following.

$J=1$ (only κ is measured):

$$\begin{aligned}
&\tilde{\tilde{W}}_{\kappa}(\theta',\phi';\theta'',\phi'')\\
&= \frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \frac{1}{4\pi}\bigg[Y_{00}(\theta'')Y_{00}(\theta') + \frac{1}{2\sqrt{5}}\left(2|A_0|^2-|A_1|^2\right)\\
&\quad\times\left(2|E_0|^2-|E_1|^2\right)Y_{00}(\theta'')Y_{20}(\theta')\bigg]\\
&\quad+\frac{\sqrt{3}}{4\pi}\varepsilon_2\bigg\{-\frac{1}{\sqrt{10}}\left(|E_0|^2-2|E_1|^2\right)Y_{20}(\theta'')Y_{00}(\theta')\\
&\quad-\frac{1}{5\sqrt{2}}\left(2|A_0|^2-|A_1|^2\right)Y_{20}(\theta'')Y_{20}(\theta')\\
&\quad-\frac{3}{5\sqrt{2}}\left(2|A_0|^2-|A_1|^2\right)\mathrm{Re}(E_1E_0^*)\\
&\quad\times\mathrm{Re}[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')]\\
&\quad-\frac{3}{5\sqrt{2}}\left(2|A_0|^2-|A_1|^2\right)\mathrm{Im}(E_1E_0^*)\\
&\quad\times\mathrm{Im}[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')]\bigg\}. \qquad (88)
\end{aligned}$$

$J=1$ (only α_1 is measured):

$$\begin{aligned}
&\tilde{\tilde{W}}_{\alpha_1}(\theta',\phi';\theta'',\phi'')\\
&= \frac{1}{4}\sum_{\kappa}^{\pm 1}\sum_{\mu}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \frac{1}{4\pi}\bigg[Y_{00}(\theta'')Y_{00}(\theta') + \frac{1}{2\sqrt{5}}\left(2|A_0|^2-|A_1|^2\right)\\
&\quad\times\left(2|E_0|^2-|E_1|^2\right)Y_{00}(\theta'')Y_{20}(\theta')\bigg]\\
&\quad+\frac{1}{4\pi}\varepsilon_1\bigg\{\frac{3}{\sqrt{10}}\left(2|A_0|^2-|A_1|^2\right)\mathrm{Im}(E_1E_0^*)\\
&\quad\times\mathrm{Im}[Y_{11}(\theta'',\phi'')Y_{21}(\theta',\phi')]\bigg\}\\
&\quad+\frac{\sqrt{3}}{4\pi}\varepsilon_2\bigg\{-\frac{1}{\sqrt{10}}\left(|E_0|^2-2|E_1|^2\right)Y_{20}(\theta'')Y_{00}(\theta')\\
&\quad-\frac{1}{5\sqrt{2}}\left(2|A_0|^2-|A_1|^2\right)Y_{20}(\theta'')Y_{20}(\theta')\\
&\quad-\frac{3}{5\sqrt{2}}\left(2|A_0|^2-|A_1|^2\right)\mathrm{Re}(E_1E_0^*)\\
&\quad\times\mathrm{Re}[Y_{21}(\theta'',\phi'')Y_{21}(\theta',\phi')]\bigg\}. \qquad (89)
\end{aligned}$$

Inspection of (88) and (89) shows that we can determine the relative magnitudes of the A and the E helicity amplitudes as well as the relative phase between E_0 and E_1 when we measure the combined angular distribution of γ_2 and e^- and the polarization of either one of the particles for the $J=1$ case. It should be noted that the measurement of the polarization of one of the two particles is essential for getting the sine of the relative phase between the two independent E helicity amplitudes.

$J=2$ (only κ is measured):

$$\begin{aligned}
&\tilde{\tilde{W}}_{\kappa}(\theta',\phi';\theta'',\phi'')\\
&= \frac{1}{4\pi}\bigg\{Y_{00}(\theta'')Y_{00}(\theta') + \frac{\sqrt{5}}{14}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\\
&\quad\times\left(2|E_0|^2+|E_1|^2-2|E_2|^2\right)Y_{00}(\theta'')Y_{20}(\theta')\\
&\quad+\frac{1}{42}\left(6|A_0|^2-4|A_1|^2+|A_2|^2\right)\\
&\quad\times\left(6|E_0|^2-4|E_1|^2+|E_2|^2\right)Y_{00}(\theta'')Y_{40}(\theta')\bigg\}\\
&\quad+\frac{\sqrt{3}}{4\pi}\varepsilon_2\bigg\{-\frac{1}{\sqrt{10}}\left(|E_0|^2-2|E_1|^2+|E_2|^2\right)Y_{20}(\theta'')Y_{00}(\theta')\\
&\quad-\frac{1}{7\sqrt{2}}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\\
&\quad\times\left(|E_0|^2-|E_1|^2-|E_2|^2\right)Y_{20}(\theta'')Y_{20}(\theta')\\
&\quad-\frac{\sqrt{3}}{7\sqrt{2}}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\\
&\quad\times\left[\mathrm{Re}(E_1E_0^*)-\sqrt{6}\,\mathrm{Re}(E_2E_1^*)\right]\\
&\quad\times\mathrm{Re}[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')]\\
&\quad-\frac{\sqrt{3}}{7\sqrt{2}}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\\
&\quad\times\left[\mathrm{Im}(E_1E_0^*)-\sqrt{6}\,\mathrm{Im}(E_2E_1^*)\right]\\
&\quad\times\mathrm{Im}[Y_{2\kappa}(\theta'',\phi'')Y_{2\kappa}(\theta',\phi')]\\
&\quad+\frac{2\sqrt{3}}{7}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\mathrm{Re}(E_2E_0^*)\\
&\quad\times\mathrm{Re}[Y_{2,2\kappa}(\theta'',\phi'')Y_{2,2\kappa}(\theta',\phi')]\\
&\quad+\frac{2\sqrt{3}}{7}\left(2|A_0|^2+|A_1|^2-2|A_2|^2\right)\mathrm{Im}(E_2E_0^*)\\
&\quad\times\mathrm{Im}[Y_{2,2\kappa}(\theta'',\phi'')Y_{2,2\kappa}(\theta',\phi')]\\
&\quad-\frac{1}{42\sqrt{10}}\left(6|A_0|^2-4|A_1|^2+|A_2|^2\right)\\
&\quad\times\left(6|E_0|^2+8|E_1|^2+|E_2|^2\right)Y_{20}(\theta'')Y_{40}(\theta')
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{7\sqrt{6}}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big[\sqrt{6}\,\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*)\big]\\
&\times\mathrm{Re}[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')]\\
&-\frac{1}{7\sqrt{6}}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big[\sqrt{6}\,\mathrm{Im}(E_1E_0^*)+\mathrm{Im}(E_2E_1^*)\big]\\
&\times\mathrm{Im}[Y_{2\kappa}(\theta'',\phi'')Y_{4\kappa}(\theta',\phi')]\\
&-\frac{1}{7}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\,\mathrm{Re}(E_2E_0^*)\\
&\times\mathrm{Re}[Y_{2,2\kappa}(\theta'',\phi'')Y_{4,2\kappa}(\theta',\phi')]\\
&-\frac{1}{7}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\,\mathrm{Im}(E_2E_0^*)\\
&\times\mathrm{Im}[Y_{2,2\kappa}(\theta'',\phi'')Y_{4,2\kappa}(\theta',\phi')]\bigg\}\,. \qquad (90)
\end{aligned}
$$

$J=2$ (only α_1 is measured):

$$
\begin{aligned}
&\tilde{\tilde{W}}_{\alpha_1}(\theta',\phi';\theta'',\phi'')\\
&=\frac{1}{4\pi}\bigg\{Y_{00}(\theta'')Y_{00}(\theta')+\frac{\sqrt{5}}{14}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big)\\
&\times\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)Y_{00}(\theta'')Y_{20}(\theta')\\
&+\frac{1}{42}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)Y_{00}(\theta'')Y_{40}(\theta')\bigg\}\\
&+\frac{\sqrt{3}}{4\pi}\varepsilon_1\bigg\{\frac{\sqrt{5}}{7\sqrt{2}}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big)\\
&\times\big[\mathrm{Im}(E_1E_0^*)+\sqrt{6}\,\mathrm{Im}(E_2E_1^*)\big]\\
&\times\mathrm{Im}[Y_{11}(\theta'',\phi'')Y_{21}(\theta',\phi')]\\
&+\frac{\sqrt{5}}{21\sqrt{2}}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big[\sqrt{6}\,\mathrm{Im}(E_1E_0^*)-\mathrm{Im}(E_2E_1^*)\big]\\
&\times\mathrm{Im}[Y_{11}(\theta'',\phi'')Y_{41}(\theta',\phi')]\bigg\}\\
&+\frac{\sqrt{3}}{4\pi}\varepsilon_2\bigg\{-\frac{1}{\sqrt{10}}\big(|E_0|^2-2|E_1|^2+|E_2|^2\big)Y_{20}(\theta'')Y_{00}(\theta')\\
&-\frac{1}{7\sqrt{2}}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big)\\
&\times\big(|E_0|^2-|E_1|^2-|E_2|^2\big)Y_{20}(\theta'')Y_{20}(\theta')\\
&-\frac{\sqrt{3}}{7\sqrt{2}}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big)\\
&\times\big[\mathrm{Re}(E_1E_0^*)-\sqrt{6}\,\mathrm{Re}(E_2E_1^*)\big]\\
&\times\mathrm{Re}[Y_{21}(\theta'',\phi'')Y_{21}(\theta',\phi')]\\
&+\frac{2\sqrt{3}}{7}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big)\,\mathrm{Re}(E_2E_0^*)\\
&\times\mathrm{Re}[Y_{22}(\theta'',\phi'')Y_{22}(\theta',\phi')]\\
&-\frac{1}{42\sqrt{10}}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big(6|E_0|^2+8|E_1|^2+|E_2|^2\big)Y_{20}(\theta'')Y_{40}(\theta')\\
&-\frac{1}{7\sqrt{6}}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\\
&\times\big[\sqrt{6}\,\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*)\big]\\
&\times\mathrm{Re}[Y_{21}(\theta'',\phi'')Y_{41}(\theta',\phi')]\\
&-\frac{1}{7}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big)\,\mathrm{Re}(E_2E_0^*)\\
&\times\mathrm{Re}[Y_{22}(\theta'',\phi'')Y_{42}(\theta',\phi')]\bigg\}\,. \qquad (91)
\end{aligned}
$$

Using (90) or (91) we can then determine the relative magnitudes of the A and the E helicity amplitudes as well as the relative phases among the E helicity amplitudes for the $J=2$ case. As in the $J=1$ case, the measurement of the polarization of one of the two particles is essential for getting the sines of the relative phases among the E helicity amplitudes.

Case 5: We first integrate over (θ',ϕ') and then average over the polarizations of γ_2. The polarizations and combined angular distribution of γ_1 and e^- are measured. Since we find that we cannot get any useful information from this polarized angular distribution for $J=1$ and $J=2$, we do not provide our results here.

Case 6: We will integrate over (θ'',ϕ'') and average over the polarizations of the electron. The combined angular distribution of the two photons and the polarization of either one of them are measured. We obtain

$J=1$ (only μ is measured):

$$
\begin{aligned}
&\tilde{\tilde{W}}_{\mu}(\theta,\phi;\theta',\phi')\\
&=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\kappa}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta''\,\mathrm{d}\theta''\,\mathrm{d}\phi''\\
&=\frac{1}{4\pi}\bigg[Y_{00}(\theta')Y_{00}(\theta)+\frac{1}{2\sqrt{5}}\big(2|A_0|^2-|A_1|^2\big)\\
&\times\big(2|E_0|^2-|E_1|^2\big)Y_{20}(\theta')Y_{00}(\theta)\bigg]\\
&-\frac{1}{4\sqrt{5}\pi}\big(2|B_0|^2-|B_1|^2\big)\\
&\times\bigg\{\big(|A_0|^2-2|A_1|^2\big)Y_{00}(\theta')Y_{20}(\theta)\\
&+\frac{1}{\sqrt{5}}\big(2|E_0|^2-|E_1|^2\big)Y_{20}(\theta')Y_{20}(\theta)\\
&+\frac{3}{\sqrt{5}}\big(2|E_0|^2-|E_1|^2\big)\,\mathrm{Re}(A_1A_0^*)\\
&\times\mathrm{Re}[Y_{2\mu}(\theta',\phi')Y^*_{2\mu}(\theta,\phi)]\\
&-\frac{3}{\sqrt{5}}\big(2|E_0|^2-|E_1|^2\big)\,\mathrm{Im}(A_1A_0^*)\\
&\times\mathrm{Im}[Y_{2\mu}(\theta',\phi')Y^*_{2\mu}(\theta,\phi)]\bigg\}\,. \qquad (92)
\end{aligned}
$$

$J=1$ (only κ is measured):

$$
\begin{aligned}
&\tilde{\tilde{W}}_{\kappa}(\theta,\phi;\theta',\phi')\\
&=\frac{1}{4}\sum_{\alpha_1}^{\pm\frac{1}{2}}\sum_{\mu}^{\pm 1}\int W_{\mu\kappa\alpha_1}(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta''\,\mathrm{d}\theta''\,\mathrm{d}\phi''
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\Big[Y_{00}(\theta')Y_{00}(\theta) + \frac{1}{2\sqrt{5}}\big(2|A_0|^2-|A_1|^2\big) \\
&\quad \times \big(2|E_0|^2-|E_1|^2\big)Y_{20}(\theta')Y_{00}(\theta)\Big] \\
&\quad - \frac{1}{4\sqrt{5}\pi}\big(2|B_0|^2-|B_1|^2\big) \\
&\quad \times \Big\{\big(|A_0|^2-2|A_1|^2\big)Y_{00}(\theta')Y_{20}(\theta) \\
&\quad + \frac{1}{\sqrt{5}}\big(2|E_0|^2-|E_1|^2\big)Y_{20}(\theta')Y_{20}(\theta) \\
&\quad - 3(-1)^{\frac{1}{2}(1-\kappa)}|E_1|^2\,\mathrm{Im}(A_1A_0^*)\,\mathrm{Im}[Y_{11}(\theta',\phi')Y_{21}^*(\theta,\phi)] \\
&\quad + \frac{3}{\sqrt{5}}\big(2|E_0|^2-|E_1|^2\big)\,\mathrm{Re}(A_1A_0^*) \\
&\quad \times \mathrm{Re}[Y_{21}(\theta',\phi')Y_{21}^*(\theta,\phi)]\Big\}. \qquad (93)
\end{aligned}
$$

$J=2$ (only μ is measured):

$$
\begin{aligned}
&\tilde{\tilde{W}}_\mu(\theta,\phi;\theta',\phi') \\
&= \frac{1}{4\pi}\Big\{Y_{00}(\theta')Y_{00}(\theta) + \frac{\sqrt{5}}{14}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big) \\
&\quad \times \big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)Y_{20}(\theta')Y_{00}(\theta) \\
&\quad + \frac{1}{42}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big) \\
&\quad \times \big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)Y_{40}(\theta')Y_{00}(\theta)\Big\} \\
&\quad - \frac{1}{4\sqrt{5}\pi}\big(2|B_0|^2-|B_1|^2\big) \\
&\quad \times \Big\{\big(|A_0|^2-2|A_1|^2+|A_2|^2\big)Y_{00}(\theta')Y_{20}(\theta) \\
&\quad + \frac{\sqrt{5}}{7}\big(|A_0|^2-|A_1|^2-|A_2|^2\big) \\
&\quad \times \big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)Y_{20}(\theta')Y_{20}(\theta) \\
&\quad + \frac{\sqrt{15}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big) \\
&\quad \times \big[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)\big]\,\mathrm{Re}[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)] \\
&\quad - \frac{\sqrt{15}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big) \\
&\quad \times \big[\mathrm{Im}(A_1A_0^*)-\sqrt{6}\,\mathrm{Im}(A_2A_1^*)\big]\,\mathrm{Im}[Y_{2\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)] \\
&\quad - \frac{2\sqrt{30}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)\,\mathrm{Re}(A_2A_0^*) \\
&\quad \times \mathrm{Re}[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)] \\
&\quad + \frac{2\sqrt{30}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)\,\mathrm{Im}(A_2A_0^*) \\
&\quad \times \mathrm{Im}[Y_{2,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)] \\
&\quad + \frac{1}{42}\big(6|A_0|^2+8|A_1|^2+|A_2|^2\big) \\
&\quad \times \big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)Y_{40}(\theta')Y_{20}(\theta) \\
&\quad + \frac{\sqrt{5}}{7\sqrt{3}}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big) \\
&\quad \times \big[\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)\big]\,\mathrm{Re}[Y_{4\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)] \\
&\quad - \frac{\sqrt{5}}{7\sqrt{3}}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big) \\
&\quad \times \big[\sqrt{6}\,\mathrm{Im}(A_1A_0^*)+\mathrm{Im}(A_2A_1^*)\big]\,\mathrm{Im}[Y_{4\mu}(\theta',\phi')Y_{2\mu}^*(\theta,\phi)] \\
&\quad + \frac{\sqrt{10}}{7}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)\,\mathrm{Re}(A_2A_0^*) \\
&\quad \times \mathrm{Re}[Y_{4,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)] \\
&\quad - \frac{\sqrt{10}}{7}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)\,\mathrm{Im}(A_2A_0^*) \\
&\quad \times \mathrm{Im}[Y_{4,2\mu}(\theta',\phi')Y_{2,2\mu}^*(\theta,\phi)]\Big\}. \qquad (94)
\end{aligned}
$$

$J=2$ (only κ is measured):

$$
\begin{aligned}
&\tilde{\tilde{W}}_\kappa(\theta,\phi;\theta',\phi') \\
&= \frac{1}{4\pi}\Big\{Y_{00}(\theta')Y_{00}(\theta) + \frac{\sqrt{5}}{14}\big(2|A_0|^2+|A_1|^2-2|A_2|^2\big) \\
&\quad \times \big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)Y_{20}(\theta')Y_{00}(\theta) \\
&\quad + \frac{1}{42}\big(6|A_0|^2-4|A_1|^2+|A_2|^2\big) \\
&\quad \times \big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)Y_{40}(\theta')Y_{00}(\theta)\Big\} \\
&\quad - \frac{1}{4\sqrt{5}\pi}\big(2|B_0|^2-|B_1|^2\big) \\
&\quad \times \Big\{\big(|A_0|^2-2|A_1|^2+|A_2|^2\big)Y_{00}(\theta')Y_{20}(\theta) \\
&\quad + \frac{\sqrt{5}}{7}\big(|A_0|^2-|A_1|^2-|A_2|^2\big) \\
&\quad \times \big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)Y_{20}(\theta')Y_{20}(\theta) \\
&\quad - (-1)^{\frac{1}{2}(1-\kappa)}\big(|E_1|^2+2|E_2|^2\big) \\
&\quad \times \big[\sqrt{3}\,\mathrm{Im}(A_1A_0^*)-\sqrt{2}\,\mathrm{Im}(A_2A_1^*)\big] \\
&\quad \times \mathrm{Im}[Y_{11}(\theta',\phi')Y_{21}^*(\theta,\phi)] \\
&\quad + \frac{\sqrt{15}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big) \\
&\quad \times \big[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)\big]\,\mathrm{Re}[Y_{21}(\theta',\phi')Y_{21}^*(\theta,\phi)] \\
&\quad - \frac{2\sqrt{30}}{7}\big(2|E_0|^2+|E_1|^2-2|E_2|^2\big)\,\mathrm{Re}(A_2A_0^*) \\
&\quad \times \mathrm{Re}[Y_{22}(\theta',\phi')Y_{22}^*(\theta,\phi)] \\
&\quad - \frac{\sqrt{3}}{\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\big(2|E_1|^2-|E_2|^2\big) \\
&\quad \times \big[\sqrt{2}\,\mathrm{Im}(A_1A_0^*)+\sqrt{3}\,\mathrm{Im}(A_2A_1^*)\big] \\
&\quad \times \mathrm{Im}[Y_{31}(\theta',\phi')Y_{21}^*(\theta,\phi)] \\
&\quad - \frac{\sqrt{30}}{\sqrt{7}}(-1)^{\frac{1}{2}(1-\kappa)}\big(2|E_1|^2-|E_2|^2\big)\,\mathrm{Im}(A_2A_0^*) \\
&\quad \times \mathrm{Im}[Y_{32}(\theta',\phi')Y_{22}^*(\theta,\phi)] \\
&\quad + \frac{1}{42}\big(6|A_0|^2+8|A_1|^2+|A_2|^2\big) \\
&\quad \times \big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)Y_{40}(\theta')Y_{20}(\theta) \\
&\quad + \frac{\sqrt{5}}{7\sqrt{3}}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big) \\
&\quad \times \big[\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)\big]\,\mathrm{Re}[Y_{41}(\theta',\phi')Y_{21}^*(\theta,\phi)]
\end{aligned}
$$

$$+\frac{\sqrt{10}}{7}\big(6|E_0|^2-4|E_1|^2+|E_2|^2\big)\,\mathrm{Re}(A_2A_0^*)$$
$$\times\,\mathrm{Re}[Y_{42}(\theta',\phi')Y_{22}^*(\theta,\phi)]\Big\}\;. \quad (95)$$

Since the relative magnitudes of the A and the E helicity amplitudes have been obtained from the partially integrated angular distribution in case 4, an examination of (92)–(95) shows that we can also determine the relative magnitudes of the B helicity amplitudes as well as the relative phases among the A helicity amplitudes when the simultaneous angular distribution of γ_1 and γ_2 is measured with either one of their polarizations for both the $J=1$ and $J=2$ cases.

4 Concluding remarks

We have derived three model-independent expressions for the combined angular distribution of the final electron and the two gamma photons in the cascade process, $\bar{p}p\to\psi'\to\chi_J+\gamma_1\to\psi+\gamma_2+\gamma_1\to e^++e^-+\gamma_2+\gamma_1$ $(J=0,1,2)$, when $\bar{p}$ and p are unpolarized and the polarization of any one of the three decay particles is measured. Our expressions are based only on the general principles of quantum mechanics and the symmetry of the problem. We have also derived the partially integrated angular distribution functions, which give the angular distributions of γ_1, γ_2 and e^- alone and of (γ_1,γ_2) and (γ_2,e^-) with the measurement of the polarization of one particle in each case. Once these angular distributions are experimentally measured, our expressions can be used to extract all the independent helicity amplitudes in the processes, $\bar{p}p\to\psi'$, $\psi'\to\chi_J+\gamma_1$, $\chi_J\to\psi+\gamma_2$ and $\psi\to e^+e^-$ for all values of J. In fact, the analysis of the angular correlations in the final decay products will serve to verify the value of J for the intermediate χ state in the cascade process. The experimentally determined values of the helicity amplitudes can then be compared with the predictions of various dynamical models. The great advantage of measuring the angular distributions with the polarization of one particle is that one can get both the cosines and sines of the relative phases of the helicity amplitudes in the radiative decay processes $\psi'\to\chi_J+\gamma_1$ and $\chi_J\to\psi+\gamma_2$. This is very important because the helicity amplitudes are in general complex. In addition, one can also get the relative magnitudes of all the helicity amplitudes in each of the sequential decay process. Therefore, by measuring the combined angular distribution of γ_1, γ_2 and e^- with the polarization of any one of the three particles, one can get complete information on the helicity amplitudes in the two radiative decay processes mentioned above. Alternatively, one can also get the same information by measuring the two-particle angular distribution of γ_2 and e^- and that of γ_1 and γ_2 with the polarization of either one of the two particles.

It is of great advantage that we express all the angular distribution functions in terms of the orthogonal functions such as the Wigner D^J functions and the spherical harmonics. Because of this feature of our results, we can get the coefficients of these functions, which are functions of the angular-momentum helicity amplitudes, by just doing a numerical integration of the measured angular distributions.

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